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# Semicontinuous solutions for Hamilton-Jacobi equations with general Hamiltonians

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## 1. Introduction

We consider the initial value problem for the Hamilton-Jacobi equation of form

$$u_t + H(x, u_x) = 0 \quad \text{in } \mathbf{R}^n \times (0, T), \tag{1a}$$

$$u(0, x) = u_0(x), \qquad x \in \mathbf{R}^n, \tag{1b}$$

where $u_t = \partial u/\partial t$ and $u_x = (\partial_{x_1} u, \cdots, \partial_{x_n} u)$, $\partial_{x_i} u = \partial u/\partial x_i$; $\infty \geq T > 0$ is a fixed number. Our main goal is to find a suitable notion of solution when $u_0$ is discontinuous. The theory of viscosity solutions initiated by Crandall and Lions [CL] yields the global solvability of the initial value problem by extending the notion of solutions when $u_0$ is continuous (cf. [E, Chap.10], [L], [B]). In fact, if initial data $u_0$ is bounded, uniformly continuous, it is well-known [CL], [L] that the initial value problem (1a)-(1b) admits a unique global (uniformly) continuous viscosity solutions when $H$ is enough regular, for example $H$ satisfies the Lipschitz conditions

$$|H(x, p) - H(x, q)| \leq C|p - q| \tag{2a}$$

$$|H(x, p) - H(y, p)| \leq C(1 + |p|)|x - y|. \tag{2b}$$

We only refer to [B], [L] and [CIL] for the basic theory of viscosity solutions. The notion of viscosity solution has been extended to semicontinuous functions. This

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is very important to prove the existence of solutions without appealing hard estimates. Such a method is first introduced by [I]. However, if $u_0$ is, for example, upper semicontinuous, a classical semicontinuous viscosity solution may not be unique.

Recently to overcome this inconvenience, Barron and Jensen [BJ] introduced another notion of viscosity solutions for semicontinuous functions when the Hamiltonian $H = H(x,p)$ is concave in $p$ and proved the existence and the uniqueness of their solution for (1a), (1b) for bounded (from above), upper semicontinuous initial data $u_0$. Their solution is now called a bilateral solution [BD]. For later development of the theory as well as other approaches we refer to [BD] and references cited there. However, their theory is limited for concave $H$. (In [BJ] $H$ is assumed to be convex but they consider the terminal value problem which is easily transformed to the initial value problem with concave Hamiltonian by setting $T-t$ by $t$.)

In this paper we introduce a new notion of a solution which is unique for a given initial upper semicontinuous initial data. For (1a), (1b) we consider auxiliary problem

$$\psi_t - \psi_y H(x, -\psi_x/\psi_y) = 0 \quad \text{in } \mathbf{R}^{n+1} \times (0,T), \tag{3a}$$

$$\psi(0,x,y) = \psi_0(x,y), \qquad (x,y) \in \mathbf{R}^n \times \mathbf{R}. \tag{3b}$$

The equation (3a) is called the level set equation for the evolution of the graph of $u$ of (1a). In fact, if a level set of a solution $\psi$ of (3a) is given as the graph of a function $v = v(t,x)$, then $v$ must solve (1a). For given upper semicontinuous initial data $u_0 : \mathbf{R}^n \to \mathbf{R} \cup \{-\infty\}$, shortly $u_0 \in$ USC $(\mathbf{R}^n)$, we take

$$\psi_0(x,y) = -\min\{\text{dist}((x,y), K_0),\ 1\}, \tag{4}$$

where

$$K_0 = \{(x,y) \in \mathbf{R}^n \times \mathbf{R};\ y \le u_0(x)\}. \tag{5}$$

We solve (3a), (3b) and set

$$\overline{u}(t,x) = \sup\{y \in \mathbf{R};\ \psi(t,x,y) \ge 0\}, \tag{6}$$

where $\psi$ is the continuous viscosity solution of (3a), (3b). We call $\overline{u}$ an *L-solution* of (1a), (1b). Such a solution uniquely exists globally in time under suitable condition on $H$.

**Theorem 1.** *Assume that the recession function*

$$H_\infty(x,p) = \lim_{\lambda\downarrow 0} \lambda H(x,p/\lambda), \quad x \in \mathbf{R}^n,\ p \in \mathbf{R}^n \tag{7}$$

*exists and that $H$ satisfies (2a), (2b). Then there exists a global unique L-solution for an arbitrary $u_0 \in USC\ (\mathbf{R}^n)$.*

One may relax the assumptions on $H$ (cf. Remark right before references) but in this paper we shall always assume (2a), (2b) and (7). These assumptions guarantee that the singularity at $\psi_y = 0$ in (3a) is removable if we restrict $\psi$ satisfying $\psi_y \leq 0$. Moreover, (3a), (3b) admits a unique global solution for any bounded, uniformly continuous initial data $\psi_0 = \psi_0(x,y)$ which is nonincreasing in $y$. (The monotonicity of the solution $\psi$ in $y$ is preserved for $t > 0$.)

## 2. Comparison and uniqueness

Since a solution of (3a), (3b) enjoys a comparison principle, so does an $L$-solution (1a), (1b).

**Theorem 2 (Comparison).** *Let $u$ and $v$ be the L-solution of (1a), (1b) with initial data $u_0$ and $v_0$, respectively, where $u_0, v_0 \in USC\ (\mathbf{R}^n)$. If $u_0 \leq v_0$ on $\mathbf{R}^n$, then $u \leq v$ on $\mathbf{R}^n \times (0,T)$.*

In the definition of an $L$-solution the specific form of $\psi_0$ given by (4) is not important.

**Theorem 3 (Uniqueness).** *Assume that $\psi_0$ is a bounded uniformly continuous function such that $\{\psi_0 \geq 0\} = K_0$ and that $y \mapsto \psi_0(x,y)$ is nonincreasing. Let $\psi$ be the solution of (3a), (3b). Then*

$$\tilde{u}(t,x) = \sup\{y \in \mathbf{R};\ \psi(t,x,y) \geq 0\}, \quad t \in (0,T),\ x \in \mathbf{R}^n$$

*agrees with the L-solution of (1a), (1b).*

The key observation for the proof is that the set $\{\psi \geq 0\}(= \{(t,x,y);\ \psi(t,x,y) \geq 0\}$ depends only on $K_0$ and is independent of the choice of $\psi_0$. This is a typical uniqueness property of a level set equation. It is based on invariance of solution under the change of the dependent variable as stated below (which is slightly more general than stated in references [ESou], [ES], [CGG1], [G], [IS] since $\theta$ need not be continuous).

**Lemma 4 (Invariance).** *Assume that $\psi$ is a subsolution (resp. supersolution) of (3a). Assume that $\theta$ is upper (resp. lower) semicontinuous and nondecreasing. Assume that $\theta \not\equiv -\infty$ (resp. $\theta \not\equiv +\infty$). Then the composite function $\theta \circ \psi$ is also a subsolution (resp. supersolution of (3a)).*

If $\{\psi \geq 0\}$ were a bounded set, a comparison principle for (3a), (3b) and Lemma 4 would yield the uniqueness of $\{\psi \geq 0\}$ as in [ES], [CGG1], [G]. However, since $\{\psi \geq 0\}$ is unbounded, we actually argue as in [IS] to get the uniqueness of $\{\psi \geq 0\}$.

## 3. Consistency

We shall compare other notion of solutions.

**Theorem 5.** *Let $\overline{u}$ be the L-solution of (1a), (1b) with $u_0 \in USC\ (\mathbf{R}^n)$. Then $\overline{u}$ be a viscosity solution of (1a) provided that $\overline{u}$ does not take $\pm\infty$.*

**Sketch of the proof.** Let $\psi$ be the solution of (3a), (3b) with $\psi_0$ in (4). By Lemma 4 the function $I^- \circ \psi$ is a subsolution of (3a), where $I^-(\sigma) = 0$ for $\sigma \geq 0$ and $I^-(\sigma) = -\infty$ for $\sigma < 0$. From this it is easy to see that $\overline{u}$ is a viscosity subsolution.

To prove that $\overline{u}$ is a viscosity supersolution we need to use the fact that $y \mapsto \psi(x,y)$ is nonincreasing. This implies that the lower semicontinuous envelope $(\overline{u})_*$ of

$\overline{u}$ equals

$$\underline{u}(t,x) = \inf\{y \in \mathbf{R};\ (t,x,y) \in \overline{\{\psi < 0\}}\} \quad t \in (0,T),\ x \in \mathbf{R}^n.$$

Since $I^+ \circ (\psi + 1/m)$ is a supersolution of (3a) by Lemma 4, we see, by stability as $m \to \infty$, that

$$\Psi(t,x,y) = \begin{cases} \infty & \text{for } (t,x,y) \in \operatorname{int}\{\psi \geq 0\}, \\ 0 & \text{for } (t,x,y) \in \overline{\{\psi < 0\}} \end{cases}$$

is a subsolution of (3a), where $I^+(\sigma) = 0$ for $\sigma \leq 0$ and $I^+(\sigma) = \infty$ for $\sigma > 0$. Thus $\underline{u}$ is a supersolution.

**Theorem 6.** *Assume that $u_0$ is bounded, uniformly continuous. Then the bounded, uniformly continuous viscosity solution $u$ of (1a), (1b) is an L-solution.*

This follows from Theorem 3 by choosing $\psi = ((y - u(t,x)) \wedge M) \vee M$ for $M = \sup |u|$.

**Theorem 7.** *Assume that $p \mapsto H(x,p)$ is concave. Let $\overline{u}$ be the L-solution of (3a), (3b) with $u_0 \in USC\ (\mathbf{R}^n)$ and $\sup u_0 < \infty$. Then $\overline{u}$ is a bilateral viscosity solution with initial data $u_0$.*

For the proof we use the property that the bilateral solution is given as a monotone limit of continuous viscosity solution [BJ]. Thus the proof is reduced to the next lemma.

**Lemma 8.** *Assume that $u_{0\varepsilon} \downarrow u_0 \in USC\ (\mathbf{R}^n)$ with $u_{0\varepsilon}$ which is Lipschitz in $\mathbf{R}^n$. Assume that $u_{0\varepsilon} \geq u_{0\varepsilon'} + \varepsilon - \varepsilon'$ for $\varepsilon > \varepsilon' > 0$. Let $u_\varepsilon$ be the solution of (1a), (1b) with $u_0 = u_{0\varepsilon}$. Then $\lim_{\varepsilon \to 0} u_\varepsilon$ is an L-solution of (1a), (1b) (so that it agrees with $\overline{u}$).*

The sequence $u_{0\varepsilon}$ is easily constructed by setting $u_{0\varepsilon} = u_0^\varepsilon + \varepsilon$ with sup-convolution $u_0^\varepsilon$ of $u_0$.

## 4. Right accessibility

It is not clear in what sense the initial value is attained for $L$-solutions (unless initial data is continuous.) Since the viscosity solution of (3a), (3b) with $\psi_0$ in (4) is continuous up to $t=0$, the set $\{\psi \geq 0\}$ is closed in $[0,T) \times \mathbf{R}^n \times \mathbf{R}$ so that

$$u_0(x) \geq \overline{\lim_{\substack{t\downarrow 0 \\ y \to x}}} \, \overline{u}(t,y). \tag{8}$$

However, in general it is not clear whether there is a sequence $t_m \to 0$, $y_m \to x$ such that

$$u_0(x) = \lim_{m\to\infty} \overline{u}(t_m, y_m). \tag{9}$$

We call this last property the right accessibility as in [CGG2]. Since $\overline{u}$ is upper semicontinuous in $[0,T)\times\mathbf{R}^n$, the property (9) is equivalent to $u_0(x) = (\overline{u}|_{(0,T)\times\mathbf{R}^n})_*$ $(0,x)$.

We give a simple criterion for right accessibility without mentioning its proof.

**Lemma 9.** *Assume that $F \in C(\mathbf{R}^N)$ is positively homogeneous of degree one. Let $A$ be a closed convex set in $\mathbf{R}^N$. Let $w$ be the $L$-solution of*

$$w_t + F(w_z) = 0, \quad z \in \mathbf{R}^N,\ t > 0; \quad w|_{t=0} = w_0.$$

*with $w_0(z) = 0$, $z \in A$ and* $\sup\{w_0(z);\ \mathrm{dist}\,(z,A) \geq \delta\} < 0$ for $\delta > 0$. Then

$$w(t,z) = \begin{cases} 0 & z \in A + tW_\alpha \\ < 0 & \text{otherwise.} \end{cases}$$

Here

$$W_\alpha = \{z \in \mathbf{R}^N;\ \sup_{|p|=1}(z\cdot p - \alpha(p)) \leq 0\},\ \alpha(p) = -F(-p).$$

The set $W_\alpha$ is often called the Wulff shape with respect to $\alpha$ if $\alpha$ is positive. The set $W_\alpha$ may be empty. For example if $F(p) = |p|$, then $W_\alpha = \emptyset$. Thus if we consider (1a), (1b)with $H(p) = |p|$ and $u_0(x) = 0$, $x = 0$; $u_0(x) = -\infty$, $x \neq 0$, then the $L$-solution $u(t,x) = -\infty$ for all $t > 0$. Thus (9) is not fulfilled.

**Theorem 10.** *If $H$ is homogeneous degree of one, and independent of $x$, then an $L$-solution is right accessible for any $u_0 \in USC\,(\mathbf{R}^n)$ if and only if $W_\alpha \neq \emptyset$.*

**Remark 11.** Our results up to §3 can be generalized for more general equation

$$u_t + H(x, u, u_x) = 0,$$

when $H$ fulfills

(i) $H \in C(\mathbf{R}^n \times \mathbf{R} \times \mathbf{R}^n)$ and $H_\infty$ exists;

(ii) There exists a modulus $m_1$ that satisfies

$$|qH(x, y - p/q) - qH(x', y', -p/q)| \leq m_1((|x - x'| + |y - y'|)(|p| + |q| + 1);$$

(iii) For each $C_1 > 0$ there exists a modulus $m_2$ such that

$$|qH(x, y - p/q) - q'H(x, y, -p'/q')| \leq m_2(|p - p'| + |q - q'|)$$

for all $x \in \mathbf{R}^n$, $y \in \mathbf{R}$, $p, p' \in \mathbf{R}^n$, $q, q' < 0$ satisfying $|p|, |p'|, |q|, |q'| \leq C_1$;

(iv) $y \mapsto H(x, y, p)$ is nondecreasing.

A typical example of $H$ satisfying these assumptions is $a(x)\sqrt{b + |p|^\beta}$ and $a$ is Lipschitz and $0 \leq \beta \leq 1$, $b \geq 0$.

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